Restricted geometric Langlands

$k=\overline{k}$

X = smooth connected projective curve $/k$

G = connected reductive group $/k$

char $k=0$: de Rham geometric Langlands equivalence

DG cat. of algebraic D-modules

DG cat. of ind-coherent sheaves

(derived) stack of $\check{G}$-bundles on X w/ flat connection

$$\text{D-mod}(\text{Bun}_G) \cong \text{IndCoh}_{\text{Nilp}}(\text{LS}^{dR}_{\check{G}})$$

moduli stack of G-bundles on X

singular support contained in $\text{Nilp} \subset H^{-1}(T^*(\text{LS}^{dR}_{\check{G}}))$

Langlands dual group

NB:

- both DG categories are k-linear
- Bun_G and $\text{LS}^{dR}_{\check{G}}$ are defined $/k$

Conjectured by Arinkin-Gaitsgory following Beilinson-Drinfeld

Proof in progress by ABCGFLRR

$k = \mathbb{C}$: Betti geometric Langlands equivalence

DG cat of all (not necessarily ind-constructible) sheaves of $\mathbb{Q}$-vector spaces in the analytic topology

$$\mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \cong \mathrm{Ind}\,\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})$$

singular support contained in $\mathrm{Nilp} \subset T^*\mathrm{Bun}_G$

derived moduli stack of $\check{G}$-local systems on $X(\mathbb{C})$ w/ $\mathbb{Q}$-coefficients

NB:

- both DG categories are $\mathbb{Q}$-linear
- Bun_G is defined $/\mathbb{C}$, $\mathrm{LS}^{\mathrm{Betti}}_{\check{G}}$ is defined $/\mathbb{Q}$
- $\mathrm{LS}^{\mathrm{Betti}}_{\check{G}}$ depends only on the homotopy type of $X(\mathbb{C})$

Conjectured by Ben-Zvi – Nadler

Follows from the de Rham case using Riemann-Hilbert correspondence

$(\mathrm{LS}^{\mathrm{dR}}_{\check{G}})^{\mathrm{an}} \cong (\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})^{\mathrm{an}}$ as $\mathbb{C}$-analytic stacks, but generally $\mathrm{LS}^{\mathrm{dR}}_{\check{G}} \not\cong \mathrm{LS}^{\mathrm{Betti}}_{\check{G}}$ algebraically.

Q: Is there an étale version of the above, where $k=\bar{k}$ is arbitrary and we use étale $\overline{\mathbb{Q}}_\ell$-sheaves?

$k=\overline{\mathbb{F}}_q$, X_0, G_0 defined over $\mathbb{F}_q$

$\rightsquigarrow$ arithmetic Langlands conjecture for G_0/ field of rational functions on X_0

Idea (Drinfeld): automorphic forms arise as the Frobenius trace of automorphic sheaves, i.e., objects of $\mathrm{Shv}(\mathrm{Bun}_G)^{\mathrm{Frob}}$. – Weil sheaves

Back to arbitrary $k=\bar{k}$:
AGKRRV conjecture the following "restricted geometric Langlands equivalence":

DG cat. of ind-constructible étale sheaves w/ $\overline{\mathbb{Q}}_\ell$-coeff.

derived stack of étale $\check{G}$-local systems w/ "restricted variation" ($\overline{\mathbb{Q}}_\ell$-coeff.)

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \cong \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$$

singular support contained in $\mathrm{Nilp} \subset T^*\mathrm{Bun}_G$

In this case
- both DG categories are $\overline{\mathbb{Q}}_\ell$-linear
- Bun_G is defined $/k$, $\mathrm{LS}_{\check{G}}^{\mathrm{restr}}$ is defined $/\overline{\mathbb{Q}}_\ell$

The moduli stack $LS^{restr}_{\check{G}}$ and the above equivalence, make sense for other sheaf theories, including D-modules and Betti sheaves.

de Rham GL $\Rightarrow$ restricted de Rham GL

Betti GL $\Longrightarrow$ restricted Betti GL

But in the étale setting, only restricted GL is available.

Let us write E for the coefficients of our sheaf theory. So

- de Rham: $E = k$
- Betti: $E = \mathbb{Q}$
- étale: $E = \overline{\mathbb{Q}}_\ell$

Key point: in all cases, $LS^{restr}_{\check{G}}(E)$ consists of all $\check{G}$-local systems. For $? =$ de Rham or Betti, there is a canonical map

$$LS^{restr}_{\check{G}} \longrightarrow LS^{?}_{\check{G}}$$

which is a closed embedding at the reduced level and bijective on E-points.

Trace of Frobenius

$\mathcal{C}$ = DG category /E,
$F: \mathcal{C} \to \mathcal{C}$ (continuous by default)

Suppose that $\mathcal{C}$ is dualizable w/r/t Lurie tensor product.

$\rightsquigarrow \mathrm{Tr}(F, \mathcal{C}): \mathrm{Vect} \xrightarrow{F} \mathrm{End}(\mathcal{C}) \cong \mathcal{C}^\vee \otimes \mathcal{C} \xrightarrow{ev} \mathrm{Vect}$
$\quad \mathrm{Tr}(F, \mathcal{C}) \in \mathrm{End}(\mathrm{Vect}) \cong \mathrm{Vect}$

Functoriality: $F_1 : \mathcal{C}_1 \to \mathcal{C}_1$, $F_2 : \mathcal{C}_2 \to \mathcal{C}_2$,
$T: \mathcal{C}_1 \to \mathcal{C}_2$ s.t. right adjoint T^R is cts
$\alpha : T \circ F_1 \to F_2 \circ T$
$\quad \rightsquigarrow \mathrm{Tr}(T, \alpha) : \mathrm{Tr}(F_1, \mathcal{C}_1) \to \mathrm{Tr}(F_2, \mathcal{C}_2)$

$\mathcal{Y}_0$ = Artin stack locally of finite type $/\mathbb{F}_q$

$\mathcal{Y} := \mathrm{Spec}\, \overline{\mathbb{F}}_q \underset{\mathrm{Spec}\, \mathbb{F}_q}{\times} \mathcal{Y}_0$

$\mathrm{Frob}: \mathcal{Y} \to \mathcal{Y}$ geometric Frobenius
$\quad \rightsquigarrow \mathrm{Frob}_* : \mathrm{Shv}(\mathcal{Y}) \xrightarrow{\sim} \mathrm{Shv}(\mathcal{Y})$

$\mathrm{Shv}(\mathcal{Y})$ is compactly generated $\Rightarrow$ dualizable

$\rightsquigarrow \mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}(\mathcal{Y})) \in \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$

$\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$ compact
$\mathcal{F} \xrightarrow{\alpha} \mathrm{Frob}_* \mathcal{F}$ (lax) Weil structure (automatic if $\mathcal{F}$ comes from $\mathrm{Shv}(\mathcal{Y}_0)$)

Take $\mathcal{C}_1 = \mathrm{Vect}$, $\mathcal{C}_2 = \mathrm{Shv}(\mathcal{Y})$, $F_1 = \mathrm{id}_{\mathrm{Vect}}$, $F_2 = \mathrm{Frob}_*$, $T(V) = V \otimes \mathcal{F}$, α lax Weil structure

$$\rightsquigarrow \mathrm{cl}(\mathcal{F}, \alpha) := \mathrm{Tr}(T, \alpha) \in \mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}(\mathcal{Y}))$$

Construction: $\exists$ natural map

compactly (= finitely) supported $\overline{\mathbb{Q}}_\ell$-valued functions

$$LT : \mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}(\mathcal{Y})) \longrightarrow \mathrm{Fun}_c(\mathcal{Y}_0(\mathbb{F}_q))$$

such that $LT(\mathrm{cl}(\mathcal{F}, \alpha))$ is the function attached to $(\mathcal{F}, \alpha)$ by the usual Frobenius trace construction.

For $\mathcal{Y}$ quasicompact:

$$y_0 : \mathrm{Spec}\, \mathbb{F}_q \to \mathcal{Y}_0 \rightsquigarrow y : \mathrm{Spec}\, \overline{\mathbb{F}}_q \to \mathcal{Y}$$

$$y^* \circ \mathrm{Frob}_* \xrightarrow{\sim} y^*$$

$\mathcal{C}_1 = \mathrm{Vect}$, $\mathcal{C}_2 = \mathrm{Shv}(\mathcal{Y})$, $F_1 = \mathrm{id}_{\mathrm{Vect}}$, $F_2 = \mathrm{Frob}_*$, $T = y^*$, $\alpha : y^* \circ \mathrm{Frob}_* \to y^*$

$$\begin{tikzcd} \mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}(\mathcal{Y})) \arrow[r, "LT"] \arrow[dr, "\mathrm{Tr}(y^*, \alpha)"'] & \mathrm{Fun}(\mathcal{Y}_0(\mathbb{F}_q)) \arrow[d, "ev_y"] \\ & \mathrm{Vect} \end{tikzcd}$$

General $\mathcal{Y}$: write $\mathcal{Y}_0 = \bigcup_i (\mathcal{Y}_i)_0$ as a filtered union of quasicompact open substacks $(\mathcal{Y}_i)_0$

$$\mathrm{Shv}(\mathcal{Y}) \cong \lim_i \mathrm{Shv}(\mathcal{Y}_i) \cong \overset{\text{w.r.t. !-pushforwards}}{\mathrm{colim}_i}\, \mathrm{Shv}(\mathcal{Y}_i)$$

$\exists$ natural isomorphism

$$\mathrm{colim}_i\, \mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}(\mathcal{Y}_i)) \cong \mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}(\mathcal{Y}))$$

$$\rightsquigarrow LT : \mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{colim}_i\, \mathrm{Fun}((\mathcal{Y}_i)_0(\mathbb{F}_q)) \cong \mathrm{Fun}_c(\mathcal{Y}_0(\mathbb{F}_q))$$

Remark: LT is usually not an isomorphism, which is closely related to the failure of

$$\mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \xrightarrow{\boxtimes} \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y})$$

to be an isomorphism. One sufficient condition is that we can write $\mathcal{Y} = \bigcup_i \mathcal{Y}_i$ as above where $\pi_0(\mathcal{Y}_i(\overline{\mathbb{F}}_q))$ is finite $\forall i$. E.g., this is the case if $\mathcal{Y} = \mathrm{Bun}_G$ and X has genus 0.

Let us specialize to the case that $\mathcal{Y} = \mathrm{Bun}_G$.

One shows that

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$$

is compactly generated ($\Rightarrow$ dualizable) and stable under Frob_*. Thus we can consider

$$\mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)).$$

<u>Theorem</u> (AGKRRV) LT restricts to an iso.

$$\mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \xrightarrow{\sim} \mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

A related statement is that

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

$$\xrightarrow{\sim} \mathrm{Shv}_{\mathrm{Nilp} \times \mathrm{Nilp}}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$$

Spectral decomposition

Recall that for any $x \in X(k)$, the geometric Satake equivalence supplies an action

$$\mathrm{Rep}(\check{G}) \curvearrowright \mathrm{Shv}(\mathrm{Bun}_G).$$ Hecke functors

More generally, using the Beilinson-Drinfeld affine Grassmannian, we can construct

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathrm{Bun}_G) \longrightarrow \mathrm{Shv}(\mathrm{Bun}_G \times X^I)$$

for any finite set I, as well as appropriate compatibilities as I varies.

<u>Theorem</u> (Nadler-Yun) The Hecke action sends

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathrm{Bun}_G)$$

into

needs to be tweaked for $X \cong \mathbb{P}^1$

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Ind}\,\mathrm{Lisse}(X) \subset \mathrm{Shv}(\mathrm{Bun}_G \times X^I).$$

In fact, a converse to this theorem holds.

<u>Theorem</u> (AGKRRV) If $\mathcal{F} \in Shv(Bun_G)$ has the property that

$$V * \mathcal{F} \in Shv(Bun_G) \otimes IndLisse(X)$$

for any $V \in Rep(\check{G})$, then $\mathcal{F} \in Shv_{Nilp}(Bun_G)$.

Corollary: Hecke eigensheaves belong to $Shv_{Nilp}(Bun_G)$

conjectured by Laumon

Nadler-Yun + abstract nonsense

$$\xrightarrow{\text{Drinfeld, AGKRRV}} QCoh(LS_{\check{G}}^{restr}) \circlearrowright Shv_{Nilp}(Bun_G)$$

Geometric observation: if $\sigma_1, \sigma_2 \in LS_{\check{G}}^{restr}(\overline{\mathbb{Q}}_\ell)$ have non-isomorphic semisimplifications, then they lie in distinct connected components.

$$QCoh(LS_{\check{G}}^{restr}) \circlearrowright Shv_{Nilp}(Bun_G)$$

$$\Rightarrow Shv_{Nilp}(Bun_G) \cong \bigoplus_{\sigma \text{ s.s.}} Shv_{Nilp}(Bun_G)_\sigma$$

AGKRRV also use this global Hecke action

- $Shv_{Nilp}(Bun_G)$ is compactly generated.
- objects of $D\text{-mod}_{Nilp}(Bun_G)$ have regular singularities
- tensor product formula for $Shv_{Nilp}(Bun_G)$ (see above)

From now on we will write

$$\mathrm{Autom} := \mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

In particular we have

$$\mathrm{Tr}(\mathrm{Frob}_X, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \cong \mathrm{Autom}.$$

$$\mathrm{Frob} \curvearrowright X \rightsquigarrow \mathrm{Frob} \curvearrowright \mathrm{LS}_{\check{G}}^{\mathrm{restr}}$$

$$\rightsquigarrow \underbrace{\mathrm{LS}_{\check{G}}^{\mathrm{arithm}}}_{\text{"stack of Langlands parameters"}} := \left(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}\right)^{\mathrm{Frob}}$$

AGKRRV show that $\mathrm{LS}_{\check{G}}^{\mathrm{arithm}}$ is a DG Artin stack almost of finite type (in particular, quasicompact). It is, however, highly derived (not eventually coconnective).

$\mathrm{Exc} := \Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arithm}}, \mathcal{O})$ "excursion operators"

$$\left\{ \begin{array}{c} \text{V. Lafforgue's} \\ \text{excursion operators} \end{array} \right\} \longrightarrow H^0(\mathrm{Exc})$$

Enhanced trace:

$\mathcal{A}$ = symmetric monoidal DG category

$F_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ symmetric monoidal

$HH_\bullet(F_{\mathcal{A}}, \mathcal{A}) := \mathcal{A} \underset{\mathcal{A} \otimes \mathcal{A}^{rev}}{\otimes} {}_{F_{\mathcal{A}}}\mathcal{A}$ ← left $\mathcal{A}$-module structure twisted by $F_{\mathcal{A}}$

$\mathcal{A}$ symmetric monoidal
$\implies HH_\bullet(F_{\mathcal{A}}, \mathcal{A})$ symmetric monoidal

$\mathcal{M}$ = dualizable $\mathcal{A}$-module

$F_{\mathcal{A}} : \mathcal{M} \to \mathcal{M}$ compatible w/ $F_{\mathcal{A}}$

$\rightsquigarrow \mathrm{Tr}^{enh}_{\mathcal{A}}(F_{\mathcal{M}}, \mathcal{M}) \in HH_\bullet(F_{\mathcal{A}}, \mathcal{A})$

If $\mathcal{A}$ is rigid, then

$$\mathrm{Tr}(F_{\mathcal{M}}, \mathcal{M}) \cong \mathrm{Hom}_{HH_\bullet(F_{\mathcal{A}}, \mathcal{A})}(\mathbb{1}, \mathrm{Tr}^{enh}_{\mathcal{A}}(F_{\mathcal{M}}, \mathcal{M})).$$

Back to our situation:

$$\mathcal{A} = \mathrm{QCoh}(LS^{restr}_{\check{G}}), \quad F_{\mathcal{A}} = \mathrm{Frob}^*$$

$$HH_\bullet(\mathrm{Frob}^*, \mathrm{QCoh}(LS^{restr}_{\check{G}})) \cong \mathrm{QCoh}((LS^{restr}_{\check{G}})^{\mathrm{Frob}})$$

$$= \mathrm{QCoh}(LS^{arithm}_{\check{G}})$$

$$\mathcal{M} = \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G), \quad F_{\mathcal{M}} = \mathrm{Frob}_*$$

$$\rightsquigarrow \mathrm{Drinf} := \mathrm{Tr}^{enh}_{\mathrm{QCoh}(LS^{restr}_{\check{G}})}(\mathrm{Frob}_*, \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G))$$

$$\curvearrowright$$

$$\mathrm{QCoh}(LS^{arithm}_{\check{G}})$$

$\mathrm{QCoh}(LS^{restr}_{\check{G}})$ is not rigid, but we do have

$$\Gamma_{co}(LS^{arithm}_{\check{G}}, \mathrm{Drinf}) \cong \mathrm{Autom}.$$

dual to
$p^*: \mathrm{Vect} \to \mathrm{QCoh}(LS^{arithm}_{\check{G}})$

So Drinf "localizes Autom onto $LS^{arithm}_{\check{G}}$."

In particular

$$\mathrm{Exc} \curvearrowright \mathrm{Drinf} \rightsquigarrow \mathrm{Exc} \curvearrowright \mathrm{Autom}$$

This refines V. Lafforgue's action.

Q: Can we extract a statement about Autom from the (conjectural) restricted GLC?

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \cong \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$$

$$\circlearrowleft$$

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$$

$$\Rightarrow \mathrm{Autom} \cong \mathrm{Tr}(\mathrm{Frob}^!, \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}))$$

$$\cong \Gamma^{\mathrm{IndCoh}}(\mathrm{LS}_{\check{G}}^{\mathrm{arithm}}, \omega)$$

not eventually connective

This would be compatible with the action of Exc on both sides. Moreover, we would have

$$\Psi : \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{arithm}}) \longrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{arithm}})$$

$$\omega \longmapsto \mathrm{Drinf}$$